

ENUMERATING ODE EQUIVALENT HOMOGENEOUS NETWORKS

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ABSTRACT. We give an alternative criterion for ODE equivalence in identical edge homogeneous coupled cell networks. This allows us to give a simple proof of Theorem 10.3 of Aquiar and Dias, which characterizes minimal identical edge homogeneous coupled cell networks. Using our criterion we give a formula for counting homogeneous coupled cell networks up to ODE equivalence. Our criterion is purely graph theoretic and makes no explicit use of linear algebra.

1. INTRODUCTION

Coupled cell networks are used to represent systems of coupled dynamical systems schematically. Such systems appear either in various biological systems. Networks of eight coupled cells modeling central pattern generators in quadrupeds can be used to recover the primary animal gaits [3, 7]. One of the important conclusions of the theory of coupled cell systems is that the network itself imposes constraints on the possible behaviors of the system even when we lack detailed knowledge of the behavior of the cells within the network. A recent application to head movement that illustrates the importance of this is [6]. For further applications see [10].

Mathematically coupled cell networks are a subclass of vertex and edge labeled directed multigraphs with loops. Vertices with the same label represent multiple copies of the same dynamical system. Edge labels represent the type of coupling. A compatibility condition is imposed that requires every vertex with a given label to receive the same set of coupling types as inputs. As with any class of graphs there is a natural notion of *isomorphic* coupled cell networks induced by bijections between the sets of vertices. This representation of coupled cell networks follows [8]. An alternative approach is outlined in [5].

Following Stewart and Golubitsky one may associate to each coupled cell network a class of ordinary differential equations that are compatible with the network structure, the class of coupled cell systems associated to a coupled cell network. As was pointed out in [8] it is possible for non-isomorphic coupled cell networks to have the same class of coupled cell systems. In this case we term the two coupled cell networks O.D.E. equivalent. We will give a full description of the coupled cell systems associated to a coupled cell network for the simple case of identical edge homogeneous coupled cell networks. For the definition in the general case see [4].

Aguiar and Dias [1] examine the structure of O.D.E. equivalence classes for such coupled cell networks. They find a collection of *canonical normal forms* - a collection of networks whose number of edges is minimal within the equivalence class. This they term the *minimal* subclass.

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In this paper we consider the simplest type of coupled cell networks, the identical edge homogeneous coupled cell networks. These are simply directed multigraphs with loops where every vertex has the same indegree. Aldosray and Stewart gave an enumeration of these networks [2] counted up to isomorphism.

Using a simpler method specific to the case of homogeneous networks we recover Theorem 10.3 of [1] which characterizes the minimal subclass in this case. Furthermore, we are able to give a recursive formula for enumerating the minimal systems with a given number of vertices and edges.

2. COUPLED CELL SYSTEMS, COUPLED CELL NETWORKS, AND O.D.E. EQUIVALENCE.

We will deal exclusively with identical edge homogeneous coupled cell networks, hereafter referred to simply as networks.

Mathematically such a network is a directed multigraph where loops are allowed and where every vertex has the same in-degree. If the constant in-degree is r we will call the network degree r . A directed multigraph consists of a set of vertices V and a multiset of edges E with elements in $V \times V$. A multiset may be thought of as a function $E : V \times V \rightarrow \mathbb{N}$; we call this function the *edge multiplicity function*. The condition that every vertex has the same in-degree, r , is then $v \in V \sum_{u \in V} E(u, v) = r$.

Given an n cell degree r network $G = (V, E)$, a choice of finite dimensional phase space $P = \mathbb{R}^d$, and a function $F : P \times P^r \rightarrow P$ such that $F(x_1; y_1, \dots, y_r)$ is invariant under all permutations of the variables y_1, \dots, y_r , we may produce a vector field on P^n . The vector field for the variable x_i associated to cell i is

$$\dot{x}_i = F(x_i; x_{j_1^{(i)}}, \dots, x_{j_r^{(i)}})$$

where $j_1^{(i)}, \dots, j_r^{(i)}$ are the source cells for the r arcs that terminate at vertex i . The complete system is

$$\begin{aligned} \dot{x}_1 &= F(x_1; x_{j_1^{(1)}}, \dots, x_{j_r^{(1)}}) \\ &\vdots \\ \dot{x}_n &= F(x_n; x_{j_1^{(n)}}, \dots, x_{j_r^{(n)}}) \end{aligned}$$

The set of such vector fields is a subset of the vector fields on P^n .

A vector field obtained from a coupled cell network G by a choice of phase space and function F is referred to as a *coupled cell system*, or an *admissible vector field*, associated to G . We may consider the class of all admissible vector fields for a given network G and phase space P . We will denote this class of vector fields by \mathfrak{X}_G^P .

Definition: Two coupled cell networks G_1 and G_2 are called *O.D.E. equivalent* if there exists a network G'_2 isomorphic to G_2 such that for all choices of phase space P

$$\mathfrak{X}_{G_1}^P = \mathfrak{X}_{G'_2}^P.$$

More prosaically, an n cell degree r_1 network G_1 and an n cell degree r_2 network G_2 are called *O.D.E. equivalent* if there exists a network G'_2 isomorphic to G_2 such that

- (1) for all choices of phase space P and function $F_1 : P \times P^{r_1} \rightarrow P$ there exists a function $F_2 : P \times P^{r_2} \rightarrow P$ such that for all vertices i

$$F_1(x_i; x_{j_1^{(i)}}, \dots, x_{j_{r_1}^{(i)}}) = F_2(x_i; x_{k_1^{(i)}}, \dots, x_{k_{r_2}^{(i)}})$$

where $j_1^{(i)}, \dots, j_{r_1}^{(i)}$ are the source cells for the r_1 arcs that terminate at cell i in network G_1 and $k_1^{(i)}, \dots, k_{r_2}^{(i)}$ are the source cells for the r_2 arcs that terminate at cell i in network G'_2 .

- (2) for all choices of phase space P and function $F_2 : P \times P^{r_2} \rightarrow P$ there exists a function $F_1 : P \times P^{r_1} \rightarrow P$ such that for all vertices i

$$F_1(x_i; x_{j_1^{(i)}}, \dots, x_{j_{r_1}^{(i)}}) = F_2(x_i; x_{k_1^{(i)}}, \dots, x_{k_{r_2}^{(i)}})$$

where $j_1^{(i)}, \dots, j_{r_1}^{(i)}$ are the source vertices for the r_1 arcs that terminate at vertex i in network G_1 and $k_1^{(i)}, \dots, k_{r_2}^{(i)}$ are the source vertices for the r_2 arcs that terminate at vertex i in network G'_2 .

If we consider $P = \mathbb{R}$ and linear functions F_1 and F_2 then we obtain the notion of *linear equivalence*. It is shown in [4] that linear equivalence and O.D.E. equivalence are equivalent.

3. NETWORK OPERATIONS THAT PRESERVE O.D.E. EQUIVALENCE

In this section we introduce two operations that can be performed on a network that preserve the O.D.E. equivalence class. Since we are dealing exclusively with homogeneous networks these operations are a small part of the network operations considered in [1]. Both Lemma 1 and Lemma 2 can be deduced from the more general arguments in [1], in particular from Proposition 7.4. For completeness we give proofs of both Lemma 1 and Lemma 2 using only what is required for our simpler case. That we may consider only these two network operations and not more general operations is crucial for the results in Section 5.

Here we give two simple operations on networks that preserve the O.D.E. equivalence class of the network.

- (1) Adding loops: A single loop is added to all vertices in the network.
- (2) k -Splitting edges: Each edge in the network is replaced by k identical copies of the edge.

Intuitively, it should be clear that these operations preserve the O.D.E. equivalence class of the network; however, a formal proof is surprisingly difficult if one does not use the notion of linear equivalence.

Lemma 1. *If network G' is obtained from network G by either of the two network operations above then G and G' are O.D.E. equivalent.*

Proof. Using [4] it is enough to prove that the two networks are equivalent when the variables x_i are taken to be in \mathbb{R} and the function F is taken to be linear. In this case we observe that for a degree r network the function F must take the form

$$F(x; y_1, \dots, y_r) = ax + b(y_1 + \dots + y_r).$$

Consider a degree r network. Adding a loop to every vertex we obtain a degree $r + 1$ network.

Given a function $F_r : \mathbb{R}^r \rightarrow \mathbb{R}$ defined by

$$F_r(x; y_1, \dots, y_r) = ax + b(y_1 + \dots + y_r).$$

we define a function $F_{r+1} : \mathbb{R}^{r+1} \rightarrow \mathbb{R}$ by

$$F_{r+1}(x; y_1, \dots, y_{r+1}) = (a - b)x + b(y_1 + \dots + y_{r+1}).$$

Clearly we have $F_{r+1}(x; x, y_1, \dots, y_r) = F_r(x; y_1, \dots, y_r)$ and consequently the linear vector fields admissible for the degree r network are a subset of the linear vector fields admissible for the $r + 1$ degree network. We can easily go the other direction. Given any function $F_{r+1} : \mathbb{R}^{r+1} \rightarrow \mathbb{R}$ of the form

$$F_{r+1}(x; y_1, \dots, y_{r+1}) = ax + b(y_1 + \dots + y_{r+1})$$

we may define a function $F_r : \mathbb{R}^r \rightarrow \mathbb{R}$ by

$$F_r(x; y_1, \dots, y_r) = (a + b)x + b(y_1 + \dots + y_r).$$

Again we have $F_r(x; y_1, \dots, y_r) = F_{r+1}(x; x, y_1, \dots, y_r)$ and consequently we see that the two networks have precisely the same set of admissible linear vector fields.

Consider a degree r network. Performing the edge splitting operation we obtain a degree $k \times r$ network. Given any function $F_r : \mathbb{R}^r \rightarrow \mathbb{R}$ of the form

$$F_r(x; y_1, \dots, y_r) = ax + b(y_1 + \dots + y_r).$$

we may define a function $F_{k \times r} : \mathbb{R}^{k \times r} \rightarrow \mathbb{R}$ by

$$F_{k \times r}(x; y_1, \dots, y_{k \times r}) = ax + \frac{b}{k}(y_1 + \dots + y_{k \times r}).$$

Clearly we have

$$(1) \quad F_{k \times r}(x; \overbrace{y_1, \dots, y_1}^{k\text{-times}}, \dots, \overbrace{y_r, \dots, y_r}^{k\text{-times}}) = F_r(x; y_1, \dots, y_r)$$

and consequently the linear vector fields admissible for the degree r network are a subset of the linear vector fields admissible for the degree $k \cdot r$ network.

Given any function $F_{k \times r} : \mathbb{R}^{k \times r} \rightarrow \mathbb{R}$ of the form

$$F_{k \times r}(x; y_1, \dots, y_{k \times r}) = ax + b(y_1 + \dots + y_{k \times r})$$

we may define a function $F_r : \mathbb{R}^r \rightarrow \mathbb{R}$ by

$$F_r(x; y_1, \dots, y_r) = ax + kb(y_1 + \dots + y_r).$$

Again equation (1) holds, and consequently we see that the two networks have precisely the same set of admissible linear vector fields.

In both cases we see that the operation produces a new network with precisely the same set of admissible linear vector fields. Thus we have that the operations preserve the O.D.E. equivalence class. \square

The operations create a network with a larger degree. However, when a network has the required structure, the inverse of these operations may be applied to produce a network with a smaller degree.

Lemma 2. *For any identical edge homogeneous coupled cell network G , there exists an O.D.E. equivalent network G_M with the following properties:*

- (1) *At least one vertex has no loops, and*
- (2) *The greatest common divisor of the multiplicities of the edges is 1.*

We will refer to G_M as a reduced network associated to G . If G is not a reduced network then G_M has a lower degree than G .

Proof. Let s denote the minimum number of loops on a vertex in G . Consider the new network G' formed by removing exactly s loops from every vertex. Clearly G' has a vertex with no loops. Since G may be obtained from G' by adding s loops we see that G and G' are O.D.E. equivalent. Let d denote the greatest common divisor of the edge multiplicities in G' . We may form a new network G_M by dividing all the edge multiplicities by d . Since G' had a vertex with no loops so does G_M . The greatest common divisor of the edge multiplicities of G_M is 1 by construction. Since we may obtain G' from G_M by splitting each edge into d edges we see that G' and G_M are O.D.E. equivalent by Lemma 1. Thus G and G_M are O.D.E. equivalent and G_M has the required properties. \square

The use of G_M to denote the reduced network is not accidental. We will now show that G_M is indeed the unique minimal network in the O.D.E. equivalence class of G . Since any network is O.D.E. equivalent to such a reduced network, it suffices to show that two reduced networks that are O.D.E. equivalent are isomorphic.

Lemma 3. *If G_1 and G_2 are reduced network, s and G_1 and G_2 are O.D.E. equivalent, then G_1 and G_2 are isomorphic.*

Proof. Let G'_2 be the network isomorphic to G_2 which appears in the definition of O.D.E. equivalence. We will show that G_1 and G'_2 are equal. If we take the phase space P for the cells to be \mathbb{R} and consider linear functions of the form $F(x, y_1, \dots, y_r) = ax + b(y_1 + \dots + y_r)$, then we see that for any choice of a_1, b_1 there must exist a_2, b_2 , and for any choice of a_2, b_2 there must exist a_1, b_1 , such that

$$(a_1 \text{Id} + b_1 A)x = (a_2 \text{Id} + b_2 B)x$$

where A is the adjacency matrix associated to G_1 , B is the adjacency matrix associated to G'_2 , and $x = (x_1, \dots, x_n)^t \in \mathbb{R}^n$. Since this holds for all $x \in \mathbb{R}^n$ we must have

$$(2) \quad a_1 \text{Id} + b_1 A = a_2 \text{Id} + b_2 B.$$

This matrix condition can be reduced to a system of linear equations of two types:

$$(3) \quad a_1 + b_1 A_{ii} = a_2 + b_2 B_{ii} \quad 1 \leq i \leq n$$

$$(4) \quad b_1 A_{ij} = b_2 B_{ij} \quad 1 \leq i, j \leq n, i \neq j$$

Since both A and B have a zero on the diagonal they must have some non-zero off diagonal entries in order to have the required row sums. Now by (4) we see that b_1 and b_2 must have the same sign and that $A_{ij} \neq 0$ if and only if $B_{ij} \neq 0$.

Since both A and B have at least one zero entry on the diagonal, either there must be an $1 \leq i \leq n$ such that $A_{ii} = B_{ii} = 0$ or there must exist $i \neq j$ such that $A_{ii} = 0$ but $B_{ii} > 0$ and $A_{jj} > 0$ but $B_{jj} = 0$. If we assume that there exists $1 \leq i, j \leq n$ with $i \neq j$ such that $A_{ii} = 0$ but $B_{ii} > 0$ and $A_{jj} > 0$ but $B_{jj} = 0$, then we obtain

$$(5) \quad a_1 = a_2 + b_2 B_{ii}$$

$$(6) \quad a_1 + b_1 A_{jj} = a_2$$

from which we immediately get $-b_1 A_{jj} = b_2 B_{ii}$ which contradicts our earlier observation that b_1 and b_2 must have the same sign. Thus there exists an $1 \leq i \leq n$ such that $A_{ii} = B_{ii} = 0$ and we can obtain from (3) that $a_1 = a_2$.

Thus we must have

$$b_1 A_{ij} = b_2 B_{ij}$$

for all $1 \leq i, j \leq n$. Now b_2 divides $b_1 A_{ij}$ for all i, j . Since the greatest common divisor of the entries of A is 1 we must have b_2 divides b_1 . Similarly b_1 divides $b_2 B_{ij}$ for all i, j . Since the greatest common divisor of the entries of B is 1, we must have b_1 divides b_2 . Since b_1 and b_2 have the same sign, we must have $b_1 = b_2$.

Finally we are able to conclude that $A = B$ so G_1 is equal to G'_2 as claimed. \square

4. EXAMPLES

First we show how Figure 1 and Figure 2 of [1] are related using our network operations.

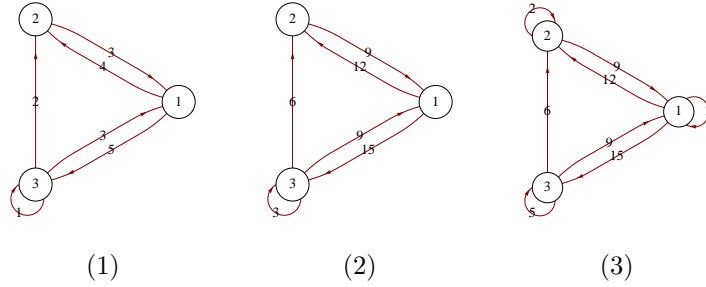


FIGURE 1. Transforming Figure 1 to Figure 2 of [1] using network operations. Edge labels represent edge multiplicities.

Referring to our Figure 1 notice that network (1) satisfies our criterion for being a minimal network. If we split each edge of network (1) into 3 edges then we obtain network (2), which is O.D.E. equivalent to network (1). If we now adjoin 2 loops to each vertex of network (2), then we obtain network (3), which is O.D.E. to network (2) and hence O.D.E. equivalent to network (1).

Next we apply the results of the previous section to the connected 3 cell degree 2 networks examined in [9]. They note that up to permutation there are 38 connected 3 cell degree 2 networks but that 8 of them are O.D.E. equivalent to the lower degree networks. Each of these 8 is obtained from one of the 4 minimal connected 3 cell degree 1 networks by either adjoining a loop to every cell or by doubling all the edges, see Figure 2.

5. ENUMERATION

We begin by outlining the work of Aldosray and Stewart in enumerating homogeneous coupled cell networks. They use the counting result known as Burnside's Lemma to enumerate all identical edge homogeneous coupled cell networks with n cells and degree r counted up to isomorphism.

To be explicit let us take $V = \{1, \dots, n\}$. Let us denote the set of all multigraphs on V with constant in-degree r by $\Omega_{n,r}$. The group of bijections on V is the symmetric group on n elements, denoted \mathbf{S}_n . Each such bijection induces a map on $\Omega_{n,r}$. Thus we have a group action of \mathbf{S}_n on $\Omega_{n,r}$. Two networks are related by \mathbf{S}_n if and only if they are isomorphic networks. Since we are counting the networks up to isomorphism what we actually want to count is the number of distinct \mathbf{S}_n orbits

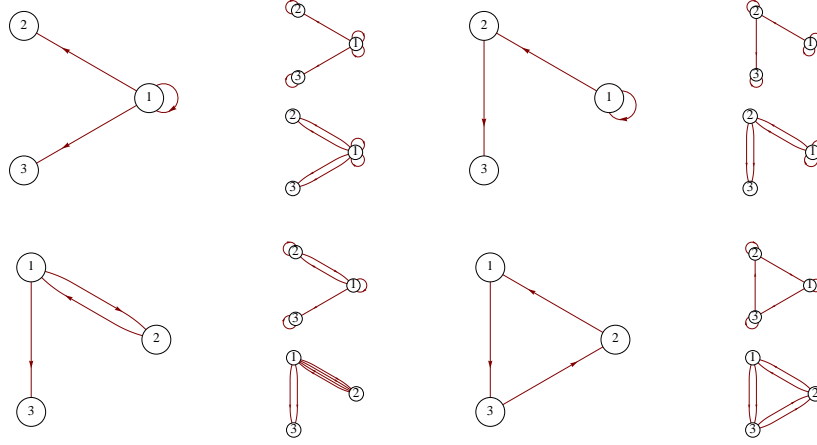


FIGURE 2. The minimal connected 3 cell degree 1 networks and their associated connected 3 cell degree 2 networks

in $\Omega_{n,r}$. Burnside's Lemma is a tool for counting the number of orbits of a group action, it states

$$|\text{Orb}_{\Omega_{n,r}}(\mathbf{S}_n)| = \frac{1}{|\mathbf{S}_n|} \sum_{g \in \mathbf{S}_n} |\text{Fix}_{\Omega_{n,r}}(g)|.$$

where $\text{Fix}_{\Omega_{n,r}}(g) = \{\omega \in \Omega_{n,r} : g \cdot \omega = \omega\}$. If g and h are conjugate elements of \mathbf{S}_n then $|\text{Fix}_{\Omega_{n,r}}(g)| = |\text{Fix}_{\Omega_{n,r}}(h)|$ and consequently we may sum over conjugacy classes rather than individual elements of \mathbf{S}_n . Suppose that $C_1 \dots, C_m$ are the conjugacy classes in \mathbf{S}_n . Let g_i be some representative of the conjugacy class C_i . We may write our sum as

$$(7) \quad |\text{Orb}_{\Omega_{n,r}}(\mathbf{S}_n)| = \frac{1}{|\mathbf{S}_n|} \sum_{i=1}^m |C_i| |\text{Fix}_{\Omega_{n,r}}(g_i)|.$$

There is a bijection between conjugacy classes of \mathbf{S}_n and partitions of the integer n . Following [2] we will denote a partition of n

$$\alpha_1 \cdot 1 + \alpha_2 \cdot 2 + \dots + \alpha_n \cdot n = n$$

by $[1^{\alpha_1} 2^{\alpha_2} \dots n^{\alpha_n}]$. The multiplicative form of this notation is perhaps unfortunate but should not cause confusion. The strength of this notation becomes apparent when we agree that if $\alpha_i = 0$ then the i^{α_i} term in the expression may be omitted. Using this notation the 7 partitions of $n = 5$ may be expressed as follows:

$5 \cdot 1$	$[1^5]$	$1 \cdot 1 + 2 \cdot 2$	$[1^1 2^2]$
$3 \cdot 1 + 1 \cdot 2$	$[1^3 2^1]$	$1 \cdot 2 + 1 \cdot 3$	$[2^1 3^1]$
$2 \cdot 1 + 1 \cdot 3$	$[1^2 3^1]$	$1 \cdot 5$	$[5^1]$
$1 \cdot 1 + 1 \cdot 4$	$[1^1 4^1]$		

The set of all partitions of n will be denoted by Π_n . An element of \mathbf{S}_5 can be associated to each $\rho \in \Pi_n$ as follows:

$$\begin{array}{llll} [1^5] & (1)(2)(3)(4)(5) & [1^1 2^2] & (1)(23)(45) \\ [1^3 2^1] & (1)(2)(3)(45) & [2^1 3^1] & (12)(345) \\ [1^2 3^1] & (1)(2)(345) & [5^1] & (12345) \\ [1^1 4^1] & (1)(2345) & & \end{array}$$

Every permutation in \mathbf{S}_5 is conjugate to one of the permutations that correspond to a partition of 5.

Every permutation $\sigma \in \mathbf{S}_n$ may be written as a product of disjoint cycles in a fashion that is unique up to the order to the cycles. The lengths of these cycles form a partition on n called the *cycle type* of the permutate σ . The permutation corresponding to a given cycle type is called the *normal form* of the cycle type. Every permutation is conjugate to the normal form of its cycle type.

Looking at the formula (7) we see that it would be advantageous to know the size of the conjugacy class associated to a given partition of n . The size of the conjugacy class corresponding to $[1^{\alpha_1} 2^{\alpha_2} \dots n^{\alpha_n}]$ is

$$(8) \quad \frac{n!}{1^{\alpha_1} 2^{\alpha_2} \dots n^{\alpha_n} \alpha_1! \alpha_2! \dots \alpha_n!}.$$

If we consider the partition determines the pattern of parentheses

$$[1^2 2^2 3^1] \quad (_) (_) (_ _) (_ _) (_ _ _)$$

then $n!$ is the number of ways of writing $1, \dots, n$ in the blanks. Observing that we can permute each cycle cyclically, that is

$$(123) \quad (231) \quad (312)$$

are all the same 3-cycle, we must factor out the $1^{\alpha_1} 2^{\alpha_2} \dots n^{\alpha_n}$ possible ways of expressing all the cycles. Finally we observe that we may permute cycles of the same length freely, so we must factor out the $\alpha_1! \alpha_2! \dots \alpha_n!$ possible orderings of the cycles.

The main difficulty in enumerating the orbits of \mathbf{S}_n lies in determining the size of the fixed point set $\text{Fix}_{\Omega_{n,r}}(g_i)$. We will give the formula for this here and refer the reader to the details in [2].

Definition: Given $\rho \in \Pi_n$ and $s \in \{1, \dots, n\}$ we may define

$$\Phi_{s,\rho}(z) = \prod_{k=1}^n (1 - z^{\frac{k}{h}})^{-\alpha_k^\rho h}$$

where $h = \gcd(s, k)$.

Clearly $\Phi_{s,\rho}(z)$ is analytic about 0 and hence we may write

$$\Phi_{s,\rho}(z) = \sum_{r=1}^{\infty} \phi_r(s, \rho) z^r.$$

Theorem 1 (Theorem 8.3 [2]). *Let $n, r \in \mathbb{N} \setminus \{0\}$. Let $H_{n,r}$ denote the number of n cell degree r networks counted up to isomorphism. $H_{n,r}$ is given by*

$$H_{n,r} = \frac{1}{n!} \sum_{\rho \in \Pi_n} \frac{n!}{1^{\alpha_1} 2^{\alpha_2} \dots n^{\alpha_n} \alpha_1! \alpha_2! \dots \alpha_n!} \prod_{k=1}^n \phi_r(k, \rho)^{\alpha_k^\rho}.$$

	r					
	1	2	3	4	5	6
1	1	1	1	1	1	1
2	3	6	10	15	21	28
n 3	7	44	180	590	1582	3724
4	19	475	6915	63420	412230	2080827
5	47	6874	444722	14072268	265076184	3405665412
6	130	126750	43242604	5569677210	355906501686	13508534834704

TABLE 1. The number of n cell degree r networks counted up to isomorphism, $H_{n,r}$.

We use this theorem to generate Table 1.

This count however includes disconnected coupled cell networks. From the perspective of dynamical systems we are interested only in the connected identical edge coupled cell networks. A disconnected system can be decomposed into a number of connected systems. Thus a disconnected n cell network corresponds to a partition of n with $\alpha_n = 0$ i.e. any partition of n except $[n^1]$.

If we denote the number of connected n cell degree r networks by $K_{n,r}$ then we may enumerate the number of disconnected coupled cell networks as follows

$$\sum_{\substack{\rho \in \Pi_n \\ \alpha_n^\rho = 0}} \prod_{m=1}^{n-1} \binom{K_{m,r} + \alpha_m^\rho - 1}{\alpha_m^\rho}$$

where

$$\binom{K_{m,r} + \alpha_m^\rho - 1}{\alpha_m^\rho}$$

is the number of ways of choosing α_m^ρ networks from the $K_{m,r}$ distinct connected m cell networks with replacement and where order does not matter. From this we obtain

Theorem 2 (Theorem 10.1 [2]). *Let $n, r \in \mathbb{N} \setminus \{0\}$. Let $K_{n,r}$ denote the number of minimal connected n cell degree r networks. We have $K_{1,r} = H_{1,r} = 1$ and for $n \geq 2$*

$$K_{n,r} = H_{n,r} - \sum_{\substack{\rho \in \Pi_n \\ \alpha_n^\rho = 0}} \prod_{m=1}^{n-1} \binom{K_{m,r} + \alpha_m^\rho - 1}{\alpha_m^\rho}.$$

We use this theorem to generate Table 2.

Now we will use the work of Section 3 to give a recursive formula for enumerating the connected minimal coupled n cell degree r networks.

Theorem 3. *Let $M_{n,r}$ denote the number of minimal connected n cell degree r networks. For $n \geq 2$ we have $M_{n,1} = K_{n,1}$ and*

$$M_{n,r} = K_{n,r} - \sum_{s=1}^{r-1} \left\lfloor \frac{r}{s} \right\rfloor M_{n,s}.$$

For $n = 1$ note that $M_{n,r} = 0$.

	r					
	1	2	3	4	5	6
1	1	1	1	1	1	1
2	2	5	9	14	20	27
n 3	4	38	170	575	1561	3696
4	9	416	6690	62725	410438	2076725
5	20	6209	436277	14000798	264632734	3403484793
6	51	117020	42722972	5554560632	355631996061	13505066262007

TABLE 2. The number of connected n cell degree r networks counted up to isomorphism, $K_{n,r}$

Proof: If a connected n cell degree r network is not minimal then it is O.D.E. equivalent to a minimal n cell s degree network where $s < n$. Given a minimal n cell degree s network G the question thus becomes how many non-isomorphic n cell degree r networks can be obtained that are O.D.E. equivalent to G . We have seen that any network G' O.D.E. equivalent to a minimal network G may be obtained from G by a combination of adjoining loops and splitting edges (and an isomorphism which we may ignore). Let A be the operation of adjoining a root and T_k the operation of k -splitting the edge. Clearly we have $T_k \circ T_l = T_{kl}$. There is a commutation relation between T_k and A , $T_k \circ A = A^k \circ T_k$. Using this commutation relation we see that any combination of adjoining loops and edge splitting can be reduced to a single k -splitting for some $k \geq 1$ followed by adjoining some number of loops. Given that G has degree s and G' has degree r the possible choices of k are constrained by $ks \leq r$. Thus there are $\lfloor r/s \rfloor$ possible values of k . We then adjoin sufficiently many loops to bring the degree to r .

The number of connected minimal n cell degree r networks is thus given by

$$M_{n,r} = K_{n,r} - \sum_{s=1}^{r-1} \left\lfloor \frac{r}{s} \right\rfloor M_{n,s}$$

with the initial condition that $M_{n,1} = K_{n,1}$ for $n \geq 2$. □

Using this theorem we generate Table 3.

	r					
	1	2	3	4	5	6
1	0	0	0	0	0	0
2	2	1	2	2	4	2
n 3	4	30	128	371	982	1973
4	9	398	6265	55628	347704	1659615
5	20	6169	430048	13558332	250631916	3138415822
6	51	116918	42605901	5511720691	350077435378	13149391543076

TABLE 3. The number of minimal connected n cell degree r networks counted up to isomorphism, $M_{n,r}$.

It is interesting to note that the number of connected minimal 2 cell degree r networks for $r \geq 2$ is given by $\phi(r)$ where ϕ is the Euler totient function. The appearance of the Euler Totient is explained by the following network diagram:

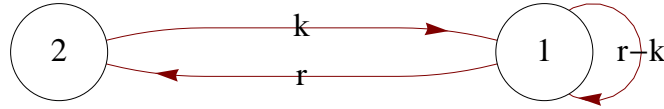


FIGURE 3. A connected 2 cell degree r network. Edge labels represent edge multiplicities.

In order for a 2 cell network to be minimal at least one vertex must have no loops. Without losing generality we may suppose that vertex 2 has no loops. Thus vertex 2 must receive r inputs from vertex 1. If we let k , with $k \leq r$, denote the number of edges from vertex 2 to vertex 1 then we see that vertex 1 must have $r - k$ loops. If this network is to be minimal then the three edge multiplicities, r , k , and $r - k$, must be relatively prime. This occurs if and only if r and k are relatively prime. For a fixed r the number of $1 \leq k \leq r$ for which r and k are relatively prime is $\phi(r)$. Provided that $r \geq 2$ we may exclude $k = 0$ since then $r - k = r$ and all edge multiplicities have divisor r and hence the network is not minimal.

If $r = 1$ then there are in fact two minimal 2 cell degree 1 networks.



FIGURE 4. The two minimal 2 cell degree 1 networks.

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